Lecture 15 Summary

PHYS798S Spring 2016
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March 28, 2016

Ginzburg Landau Theory of Superconductors
The Ginzburg-Landau (GL) theory of superconductivity is one of the most useful tools for doing quantitative calculations.

BCS theory applies to homogeneous uniform superconductors, and provides insights into the microscopic pairing mechanism. However many problems of interest involve inhomogeneities such as interfaces, impurities, magnetic vortices, etc. For this we need to generalize BCS theory. One can write the creation and annihilation operators in terms of real-space eigenfunctions, rather than momentum space eigenfunctions. An operator that annihilates a particle at location $\overrightarrow{r}$ and spin $\uparrow$ is,
$\Psi(\overrightarrow{r},\uparrow) = \sum_n \left[\gamma_{n,\uparrow} u_n(\overrightarrow{r}) - \gamma^+_{n,\downarrow} v^*_n(\overrightarrow{r})\right]$, where n labels the real space eigenfunctions. Note that the u's and v's are now position dependent. This will lead to the definition of a position-dependent gap function, determined self-consistently.

Ginzburg-Landau (GL) theory can be derived from this real-space generalized version of BCS. However we will follow the phenomenological definition of the theory.

We introduce a spatially dependent complex order parameter $\psi(\overrightarrow{r})$ which has the interpretation that the magnitude squared describes the local superfluid density, $|\psi(\overrightarrow{r})|^2 = n_s(\overrightarrow{r})$.
This order parameter shares many of the properties of the macroscopic quantum wavefunction that we discussed early in the semester as being responsible for the MQ properties such as flux quantization and the Josephson effect. It is not useful to think of the GL order parameter as a local version of the spectral gap in quasiparticle excitations.

GL proceeds by making the following postulate: ψ is "small" and varies "slowly" in space. In this case the free energy difference between the superconducting and normal states can be expanded in powers of the order parameter and its gradient as,

$f_s = f_n + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \overrightarrow{\bigtriangledown} - e^* \overrightarrow{A} \right) \psi \right|^2 + \frac{\mu_0 H^2}{2}$, where f is the free energy density (measured in J/m^3) and $\overrightarrow{A}$ is the total vector potential due to both external and self-generated fields. This expansion is expected to converge near the transition temperature where the order parameter magnitude continuously goes to zero.

Ginzburg Landau Temperature Dependence

In the absence of gradients and fields, the simple GL model reduces to,
$f_s = f_n + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4$. To keep the order parameter finite requires that β be positive. For α positive, the minimum free energy difference occurs for $\psi = 0$. If α changes sign, there is a minimum for $\psi = \psi_\infty$ such that $|\psi_\infty|^2 = -\alpha/\beta$. The free energy density of the superconductor is lower than that of the normal metal by an amount $f_s - f_n = -\frac{1}{2}\frac{\alpha^2}{\beta}$. We treat this as equivalent to a magnetic field energy density associated with the thermodynamic critical field H_c as,
$f_s - f_n = -\frac{1}{2}\frac{\alpha^2}{\beta} = -\frac{\mu_0 H_c^2}{2}$, so that $H_c^2 = \frac{\alpha^2}{\mu_0 \beta}$.

It is clear that α must change sign at T_c, hence we can write $\alpha(t) = \alpha'(t-1)$ where $t \equiv T/T_c$ is the reduced temperature and α' is positive. Putting this into the definition of ψ_∞ yields the temperature dependence of the order parameter near T_c:
$\psi_\infty^2 \propto n_s \propto (1-t)$, so that the order parameter falls continuously to zero at T_c linearly in temperature.

Ginzburg Landau Gradient Term

The gradient term is
$G = \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \overrightarrow{\bigtriangledown} - e^* \overrightarrow{A} \right) \psi \right|^2$.
Substituting $\psi = |\psi(\overrightarrow{r})| \, e^{i\phi(\overrightarrow{r})}$ and simplifying yields,
$G = \frac{1}{2m^*} \left[\hbar^2 (\bigtriangledown |\psi|)^2 + (\hbar \overrightarrow{\bigtriangledown} \phi - e^* \overrightarrow{A})^2 |\psi|^2 \right]$.
This shows that creating a gradient in the magnitude of the order parameter costs energy. Likewise creating a strong twist in the phase is also energetically costly. Recalling our earlier results for the MQWF, we can write the second term as $\frac{1}{2} m^* v_s^2 |\psi|^2$ which can be interpreted as the kinetic energy density of the superfluid flow, created by a twist in phase and vector potential.

0.1 GL Effective Penetration Depth

We now define an effective magnetic penetration depth as
$\lambda_{eff}^2 \equiv \frac{m^*}{\mu_0 |\psi|^2 (e^*)^2}$. This differs from the London penetration depth in the substitution of the effective superfluid density $|\psi|^2$ for the total electron density n. As such, the effective screening length is often longer than the London penetration depth, as we shall see later.

We can now write the two phenomenological GL parameters in terms of measurable quantities as follows,

$\alpha(T) = -\frac{2\mu_0^2 e^2}{m} H_c^2(T)\lambda_{eff}^2(T),$
$\beta(T) = \frac{4\mu_0^3 e^4}{m^2} H_c^2(T)\lambda_{eff}^4(T).$

Using the empirical temperature dependences $H_c \propto (1-t^2)$ and $\lambda_{eff}^{-2} \propto (1-t^4)$ yields the following temperature dependences near T_c:
$|\psi|^2 \propto 1-t,$
$\alpha \propto -H_c^2\lambda_{eff}^2 \propto -(1-t),$
$\beta \propto H_c^2\lambda_{eff}^4 \propto \frac{1}{(1+t^2)^2} \propto$ constant near T_c.

0.2 GL Differential Equation

The total free energy difference
$\int (f_s - f_n) d^3r$
is a functional of the GL order parameter function $\psi(r)$ and the total vector potential $A(r)$. By demanding that the free energy difference is a minimum for a first order functional variation of $\psi(r)$, one can derive the GL differential equation:
$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\overrightarrow{\bigtriangledown} - e^*\overrightarrow{A}\right)^2\psi = 0.$

Similarly for a functional variation of the vector potential one finds,
$\overrightarrow{J} = \frac{e^*}{m^*}|\psi|^2\left(\hbar\overrightarrow{\bigtriangledown}\theta - e^*\overrightarrow{A}\right)$
$= e^*|\psi|^2\overrightarrow{v}_s.$

One can view the GL differential equation as a nonlinear Schrodinger equation with a potential of the form $V = \beta|\psi|^2$ and eigenvalue $-\alpha$. Since $\beta > 0$ the potential is repulsive, tending to spread out the distribution of $\psi(\overrightarrow{r})$.

0.3 "Derivation" of GL Theory

Starting from the BCS self-consistent gap equation one can expand the integral to third order in the limit of small gap (near T_c). Generalizing the homogeneous BCS gap $\Delta(T)$ to a spatially-dependent "gap function" $\Delta(\overrightarrow{r})$ and including the possibility of a gradient in the gap, as well as magnetic fields, yields this equation (Ketterson and Song, section 45);
$\left[\left(1-\frac{T}{T_c}\right) + \frac{1}{6}\left(\frac{\hbar v_F}{\pi k_B T_c}\right)^2 \frac{7}{8}\zeta(3)\chi(\rho)\left(\overrightarrow{\nabla} - \frac{2ie}{\hbar c}\overrightarrow{A}\right)^2 - \frac{7\zeta(3))}{8(\pi k_B T_c)^2}\left|\Delta(\overrightarrow{r})\right|^2\right]\Delta(\overrightarrow{r}) = 0.$

This is equivalent to the GL differential equation with the assignments:
$\psi(\overrightarrow{r}) = \sqrt{\frac{7\zeta(3)n}{8(\pi k_B T_c)^2}}\Delta(\overrightarrow{r})$
$\alpha = -\frac{6\pi^2(k_B T_c)^2}{7\zeta(3)\epsilon_F}\left(1-\frac{T}{T_c}\right)$
$\beta = \frac{12\pi^2(k_B T_c)^2}{7\zeta(3)\epsilon_F n}$

where n is the total electron density and ϵ_F is the Fermi energy.

The gap function $\Delta(\overrightarrow{r})$ is no longer the gap in the excitation spectrum of the superconductor. Instead it can be thought of as a local internal field tending to produce Cooper pairing of electrons. It is also a complex function, in general.